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Solitons in nonlinear optics I. A more accurate description of the 2π pulse in self-induced transparency

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Abstract. A more accurate description of the 2π pulse of self-induced transparency is obtained. This is an exact solution of an approximate form of the Maxwell–Bloch equations in which only backscattering is neglected; the equations are valid at densities less than about 10^{18} atoms cm^{-3}. At such densities the 2π sech pulse of McCall and Hahn remains a good approximation well into the picosecond range. The more accurate solution is chirped by a factor proportional to the square of the ratio of the spectral width of the pulse to its carrier wave frequency. The stability of this pulse solution and other N soliton solutions of the approximate Maxwell–Bloch equations is demonstrated.

1. Introduction

This is the first in a series of papers devoted to applications of multisoliton solutions of model equations in nonlinear optics in the two level atom approximation. The subject of nonlinear optical pulse propagation in a resonant medium has recently received an excellent review by Lamb (1971). At this time only 1, 2, and 3 soliton solutions of the equations describing a slowly varying envelope modulating a resonant optical carrier were known†. These solutions (except the one soliton solution) were not inhomogeneously broadened, although Lamb (1972, 1973) has since reported a broadened form of the two soliton pulse. More recently we have published analytic expressions for N soliton 'sharp line' solutions of these equations (Caudrey *et al* 1973a) and the broadened version of these solutions has now been found (Caudrey *et al* 1973b). These solutions are particularly relevant to the theory of self-induced transparency (SIT) first put forward by McCall and Hahn (1967, 1969).

In this series of papers we use the N soliton solutions to develop a theory of SIT somewhat more general than has been available before. We consider not only the envelope equations taken by Lamb (1971) and others, but also the more basic equations of nonlinear optics for which (cf Caudrey *et al* 1973b) we have multisoliton solutions. We restrict our attention throughout to the case of EM waves propagating in an attenuator, that is, all the atoms are initially in the lower state.

In this first paper we consider the exact solutions of an approximate form of the semiclassical Maxwell–Bloch equations describing the interaction of a classical EM field with a dielectric of quantized two level atoms. The approximation considered

† Barnard (1973) has since extended the Bäcklund transformation developed by Lamb to calculate a six soliton solution.

is equivalent to the neglect of backscattering; it has been shown (Eilbeck 1972) that this approximation is good to within 1 % at densities of about 10^{18} atoms cm^{-3} and the approximation improves in proportion at lower densities. For example in the Rb vapour studied by Gibbs and Slusher (1970) the density is about 10^{12} atoms cm^{-3} and the approximation is good to one part in 10^8. This means we can safely use the equations to investigate other approximations like the slowly varying amplitude approximation or the rotating wave approximation providing these are of the order of one part in 10^6 or larger.

We give in this paper the exact N soliton solution of these approximate Maxwell–Bloch equations. Then we show in particular that the *two* soliton solution generalizes the 2π (that is the *one* soliton) solution of the more approximate envelope theory of McCall and Hahn (1969) to regions of high energy and shorter pulse duration where more approximate theories break down. The two soliton solution we use is a closed analytic expression made up of elementary functions so that we can easily find its approximations in various limits. It can be expanded as a power series in γ, the ratio of the spectral width of the pulse to its carrier wave frequency. To zero order in γ we regain the 2π hyperbolic secant pulse of McCall and Hahn modulating an on- or off-resonant carrier wave: the corrections of experimental interest such as chirping, modified pulse shape, etc, prove to be of order γ^2 (as perhaps we should expect). Since $\gamma \simeq 10^{-3}$ for picosecond pulses the corrections to the slowly varying envelope approximation are of the order one part in 10^6 well into the picosecond range.

These figures agree with other approximate estimates (Bullough 1971, Bullough and Ahmad 1971), but with increasing experimental accuracy and peak power we have some hope of eventually observing some of the features of our generalized 2π pulse.

We shall ignore *homogeneous* broadening on the grounds that effects significant at picosecond or shorter duration will not be affected by homogeneous broadening which is on a nanosecond scale. One advantage of the multisoliton approach is that the problem of *inhomogeneous* broadening becomes almost trivial (cf Caudrey *et al* 1973b). Solely for simplicity of presentation we shall not write down every equation in broadened form: the key point is that once the sharp line solution is known the broadened results can be calculated by the simple prescription described below. In order to allow complete comparison with the original work of McCall and Hahn we give some results in the full broadened form.

The result that inhomogeneous broadening has no special significance in the two level atom theories of SIT has been confirmed in a more practical manner by computer calculations (Estes *et al* 1970) and recent experimental work (Gibbs and Slusher 1972) which show no marked difference between broad and sharp line SIT. Our results here and elsewhere show that broadening only changes the individual velocity of each pulse without altering any other phenomena of physical interest. Of course inhomogeneous broadening plays an important part in other pulse phenomena such as photon echo (Lamb 1971). The area theorem of McCall and Hahn also depends on inhomogeneous broadening, but similar results can be obtained in the sharp line case, as demonstrated by Lamb.

The paper is set out as follows: in § 2 we review the basic equations of semiclassical nonlinear optics and introduce the so called reduced Maxwell–Bloch (RMB) equations valid at low densities. The N soliton solution of these equations is described. In § 3 we show that the two soliton solution of the RMB equations gives a more accurate description of the 2π SIT pulse, and exhibit the corrections to the theory of McCall and Hahn required at shorter pulse lengths and higher energies. Finally in § 4 we demonstrate the stability of the N soliton solution of the RMB equations by the use of Liapunov techniques.

2. Some nonlinear optics equations and N soliton solutions

There are several different sets of nonlinear partial differential equations describing the propagation of EM waves through a medium of two level atoms, each set being at a different level of approximation. These different sets are mathematically very similar, but each has a different physical interpretation. To add to the confusion, there is no universally accepted convention for naming these equations. We shall spend some time in the first three subsections reviewing the most important sets of equations, naming them, and discussing their similarities and differences.

2.1. The Maxwell–Bloch equations

The most basic semiclassical equations governing the propagation of EM waves in a dielectric of two level atoms are the Maxwell wave equation

$$E_{xx}(x,t)-c^{-2}E_{tt}(x,t) = 4\pi c^{-2}np\langle r_{tt}(x,t,\omega_s')\rangle \tag{2.1}$$

and the Bloch type equations

$$r_t(x,t,\omega_s') = -\omega_s' s(x,t,\omega_s') \tag{2.2a}$$

$$s_t(x,t,\omega_s') = \omega_s' r(x,t,\omega_s') + 2p\hbar^{-1}E(x,t)u(x,t,\omega_s') \tag{2.2b}$$

$$u_t(x,t,\omega_s') = -2p\hbar^{-1}E(x,t)s(x,t,\omega_s'). \tag{2.2c}$$

Our notation, detailed below, follows that of Eilbeck and Bullough (1972). The components of the Bloch vector (r, s, u) are appropriate real combinations of the elements of the atomic density matrix. Each atom is assumed to be at a different resonant frequency ω_s' with normalized probability $g(\omega_s')$, and the angular brackets in (2.1) are to be interpreted as an averaging procedure over all the possible frequencies, such that for any function $F(\omega_s')$

$$\langle F(\omega_s')\rangle = \int_0^\infty F(\omega_s')g(\omega_s')\,\mathrm{d}\omega_s'. \tag{2.3}$$

The spread of the atomic resonance frequency described by $g(\omega_s')$ is the inhomogeneous broadening. It is an empirical description of the microscopic interactions and motions (like the motions causing Doppler broadening) which appear to spread the levels of an isolated atom. In general $g(\omega_s')$ will be a function strongly peaked about ω_s, the atomic frequency of an isolated two level atom. In the case where these effects are negligible, all the atoms have exactly the same resonant frequency ω_s, and the function $g(\omega_s')$ becomes a Dirac delta function, $g(\omega_s') = \delta(\omega_s - \omega_s')$. In this case, which we call the 'sharp line case', we see from (2.3) that $\langle F(\omega_s')\rangle = F(\omega_s)$ and we can drop the primes in equation (2.2). The other parameters in equations (2.1), (2.2) have their usual meanings: $p = ex_{0s}$ is the matrix element of the dipole operator, n is the atomic dipole density, and $\hbar\omega_s'$ is the energy separation of the two levels. The subscripts x and t refer to partial differentiation with respect to the space and time coordinates.

We shall refer to the system of coupled nonlinear partial differential equations (2.1) with (2.2) as the Maxwell–Bloch (MB) equations. It is assumed throughout that the dielectric is initially in its ground state (an attenuator) so the boundary conditions are $E, r, s \to 0, u \to -1$, as $x \to \pm\infty$.

2.2. *The reduced Maxwell–Bloch equations*

It is possible to define a dimensionless constant α_s which is a measure of the coupling between the electric field and the atoms:

$$\alpha_s = 4\pi np^2(\hbar\omega_s)^{-1}. \tag{2.4}$$

At low densities such that α_s is much less than unity, the backscattered part of $E(x, t)$ can be neglected, and (2.1) can be reduced to an equation describing waves travelling to the right only. This can be proved by a simple application of characteristic theory (Eilbeck 1972). The reduced version of (2.1) is

$$E_x + c^{-1}E_t = -2\pi c^{-1} np\langle r_t\rangle. \tag{2.5}$$

For a typical value of p, 10^{-18} cgs units, (2.5) is a good approximation (to within 1%) at atomic densities $n \lesssim 10^{18}\ \mathrm{cm}^{-3}$. At lower densities the approximation improves proportionally so that at $n \lesssim 10^{12}$ we are neglecting terms of order of one part in 10^8. As we show below (and cf Bullough and Ahmad 1971), this approximation is implicit in the derivation of the SIT equations. We shall refer to the coupled equations (2.5) with (2.2) as the *reduced* Maxwell–Bloch (RMB) equations. A dimensionless form of the RMB equations is given in § 2.5. The main result of this paper is that exact solutions of the RMB equations exist which correspond to the 2π pulse of self-induced transparency. However in most treatments of SIT further approximations are made to describe the slowly varying *envelope* $\mathscr{E}(x, t)$ of the field modulating a resonant carrier wave.

2.3. *The self-induced transparency equations*

The approximations which give the envelope equations are:

(i) $r_{tt} = -\omega_s^2 r$. This is equivalent to assuming $2p\hbar^{-1}Eu \ll \omega_s r$ in the combination of (2.2*a*) and (2.2*b*)†.

(ii) $E(x, t) = \hbar p^{-1}\mathscr{E}(x, t)\cos\Phi(x, t)$ where $\Phi(x, t) = \kappa_s x - \omega_s t + \phi(x, t)$, in which the carrier wave is resonant, $c\kappa_s = \omega_s$, and $\mathscr{E}(x, t)$ and $\phi(x, t)$ are slowly varying functions of x (compared with κ_s^{-1}) and t (compared with ω_s^{-1}) such that second derivatives, etc, can be dropped.

(iii) The rotating wave approximation, that is, neglect of higher harmonics of ω_s.

Application of the approximations (i)–(iii) to *either* the MB *or* the RMB equations gives the most general resonant SIT equations (Lamb 1971):

$$\mathscr{E}_x + c^{-1}\mathscr{E}_t = \alpha'\langle P(x, t, \Delta\omega')\rangle \tag{2.6a}$$

$$P_t(x, t, \Delta\omega') = \mathscr{E}N + (\Delta\omega' + \phi_t)Q \tag{2.6b}$$

$$N_t(x, t, \Delta\omega') = -\mathscr{E}P \tag{2.6c}$$

$$Q_t(x, t, \Delta\omega') = -(\Delta\omega' + \phi_t)P \tag{2.6d}$$

$$\mathscr{E}(x, t)(\phi_x(x, t) + c^{-1}\phi_t(x, t)) = -\alpha'\langle Q(x, t, \Delta\omega')\rangle. \tag{2.6e}$$

† We are assuming also that $\omega_s \simeq \omega_s'$.

We have changed the frequency variable to $\Delta\omega' = \omega_s' - \omega_s$ so that in this variable

$$\langle F(\Delta\omega')\rangle = \int_{-\infty}^{\infty} F(\Delta\omega')g(\Delta\omega')\,\mathrm{d}\Delta\omega'. \tag{2.7}$$

For the sharp line case $g(\Delta\omega') = \delta(\Delta\omega')$ and $\langle F(\Delta\omega')\rangle = F(0)$. In the rotating wave approximation $N = u$ in (2.6). The constant α' is defined by $\alpha' = 2\pi np^2\omega_s(\hbar c)^{-1}$ and the functions P and Q are defined by

$$r(x, t) = P(x, t)\sin\Phi(x, t) + Q(x, t)\cos\Phi(x, t). \tag{2.8}$$

At this stage we make the further consistent set of assumptions that (iv) the atoms are symmetrically broadened about ω_s so that $g(\Delta\omega') = g(-\Delta\omega')$, and (v) Q is an odd function of $\Delta\omega'$. Under these assumptions $(\phi_x + c^{-1}\phi_t)$ is zero from (2.6). We now assume (vi) $(\phi_x + c^{-1}\phi_t) = 0$ implies that ϕ_x and ϕ_t are separately zero, and hence ϕ is a constant. Assumption (vi) is rather crucial and to our knowledge is not clearly made in the published literature. These assumptions lead to the following form of the envelope equations:

$$\mathscr{E}_x + c^{-1}\mathscr{E}_t = \alpha'\langle P\rangle \tag{2.9a}$$

$$P_t = \mathscr{E}N + \Delta\omega' Q \tag{2.9b}$$

$$N_t = -\mathscr{E}P \tag{2.9c}$$

$$Q_t = -\Delta\omega' P. \tag{2.9d}$$

In what follows we shall refer to the coupled equations (2.9) as the SIT equations. Note that this usage is not universal, for example Bullough and Ahmad (1971) apply the name SIT equations to those which we have called here the MB equations.

The SIT equations (2.9) have been derived under the assumption that the carrier wave was resonant. We can relax this assumption for a very limited class of solutions of (2.9), as follows. If the frequency of the carrier wave, ω_c, is $\omega_s + \Delta\omega_c$ we write ϕ in assumption (ii) above as $\phi(x, t) = -\Delta\omega_c t + \Delta\kappa_c x + \phi'(x, t)$. In equations (2.6*e*) we can take ϕ_x and ϕ_t to be constant, and hence set ϕ' equal to zero, if and only if $\langle Q(x, t, \Delta\omega')\rangle$ is a constant multiple of $\mathscr{E}$ for all x, t. Providing our solutions have this property (2.6*e*) becomes merely a dispersion relation for $\Delta\kappa_c$ and we can absorb $\Delta\omega_c$ into $\Delta\omega'$ in equations (2.6*b*) and (2.6*d*) to get the SIT equations (2.9). This is the approach implied in the work of McCall and Hahn (1969) and formulated as such by Bullough (1971) and Bullough and Ahmad (1971). However it must be emphasized that the only solutions for which the assumption about the proportionality of $\mathscr{E}$ and $\langle Q\rangle$ is correct are the distortionless or travelling wave solutions, that is those of the type $\mathscr{E}(x - Vt)$. One example of this type is the one soliton solution given below: the multisoliton solutions of the SIT equations are *not* of this type and are only valid solutions for a strictly resonant carrier wave.

The SIT equations (2.9) for the carrier wave *envelope* must not be confused with the MB and RMB equations for the electric *field*. Nevertheless there is a close mathematical correspondence between the SIT equations (2.9) and the RMB equations (2.2), (2.5). We will make use of this correspondence after briefly reviewing some soliton solutions of the SIT equations.

2.4. *Soliton solutions of the SIT equations*

The solitary wave (soliton) solution of the SIT equations is the well known '2π' pulse with *time* area 2π (McCall and Hahn 1969):

$$\mathscr{E}(x,t) = E_1 \operatorname{sech}(\omega_1 t - \kappa_1 x) \tag{2.10a}$$

where

$$\omega_1 = \tfrac{1}{2}E_1 \tag{2.10b}$$

and

$$\frac{\kappa_1}{\omega_1 c} = 1 + \left\langle \frac{4\alpha'}{E_1^2 + 4(\Delta\omega')^2} \right\rangle. \tag{2.10c}$$

Two and three soliton solutions of the sharp line SIT equations ($\Delta\omega' = 0$) were first found by Lamb (1971) and the general N soliton solution in the sharp line case by the authors (Gibbon and Eilbeck 1972, Caudrey *et al* 1973a). The N soliton solution for the inhomogeneously broadened SIT equations has now been found (Caudrey *et al* 1973b). It is

$$\mathscr{E}^2(x,t) = 4\frac{\partial^2}{\partial t^2} \ln f(x,t) \tag{2.11a}$$

$$f(x,t) = \det|M(x,t)| \tag{2.11b}$$

where the $N \times N$ determinant M_{ij} has elements

$$M_{ij} = \frac{2(E_i E_j)}{E_i + E_j}\{\exp(\theta_i) + (-1)^{i+j}\exp(-\theta_j)\} \tag{2.12a}$$

and

$$\theta_i = \omega_i t - \kappa_i x + \delta_i \tag{2.12b}$$

$$\omega_i = \tfrac{1}{2}E_i \tag{2.12c}$$

$$\frac{\kappa_i}{c\omega_i} = 1 + \left\langle \frac{4\alpha'}{E_i^2 + 4(\Delta\omega')^2} \right\rangle. \tag{2.12d}$$

The E_i and δ_i are $2N$ arbitrary constants determining the amplitude and the phase, respectively, of the ith soliton.

This completes our brief review of the SIT equations. We now proceed to the exact solutions of the RMB equations. These are more exact solutions of the physical problem since we do *not* make the approximations (i)–(vi) in assuming the RMB equations.

2.5. Soliton solutions of the RMB equations

First we introduced a dimensionless form of the RMB equations by the transformation $t' = \omega_a t$, $x' = \omega_a c^{-1} x$, $E' = 2p\omega_a^{-1}\hbar^{-1}E$, $(\omega_s')' = \omega_s'\omega_a^{-1}$ where ω_a is a typical atomic frequency such that $(\omega_s')'$ is of order unity. Applying this transformation to (2.2) and (2.5) we obtain, dropping the primes,

$$E_x + E_t = -\alpha_a\langle r_t(x,t,\omega_s')\rangle \tag{2.13a}$$

$$r_t = -\omega_s' s \tag{2.13b}$$

$$s_t = \omega_s' r + Eu \tag{2.13c}$$

$$u_t = -Es. \tag{2.13d}$$

The dimensionless constant α_a is defined in the same way as α_s in equation (2.4) with

ω_a replacing ω_s. That these equations are mathematically very similar to the SIT equations can be seen by replacing ω'_s, r, s, u by $\Delta\omega'$, Q, P, N. The main difference is that r_t appears inside the bracket in equation (2.13*a*) whereas in (2.9*a*) it is P which appears in the angular brackets. The sharp line equations are mathematically exactly equivalent. The proof of the N soliton solution of the equations in the sharp line case is given by Caudrey *et al* (1973c). A simple procedure for broadening the N soliton solution of the SIT equations is proved by Caudrey *et al* (1973b). The procedure and proof for broadening the N soliton solution of the RMB equations is completely analogous and gives the following results:

$$E(x,t) = E_1 \operatorname{sech}\left\{\tfrac{1}{2}E_1\left(t - x\left\langle 1 + \frac{\alpha_a \omega'_s}{E_1^2 + 4(\omega'_s)^2}\right\rangle\right)\right\}. \tag{2.14}$$

This is the RMB version of the single soliton solution of the full MB equations (2.1) and (2.2) found by Bullough and Ahmad (1971). These single pulse solutions are not yet of much experimental interest, as the resonant pulse (with $\frac{1}{2}E_1 = \omega_s$) is ultra intense (1000 TW cm^{-2}) and only of femtosecond duration†. However the *two* soliton solution is of more interest as we show below.

The N soliton solution of (2.12) is

$$E^2(x,t) = 4\frac{\partial^2}{\partial t^2}\ln f(x,t) \tag{2.15}$$

where f is that defined in equations (2.11*b*), (2.12) except that equation (2.12*d*) becomes

$$\frac{\kappa_i}{\omega_i} = 1 + \left\langle \frac{4\alpha_a \omega'_s}{E_i^2 + 4(\omega'_s)^2}\right\rangle. \tag{2.16}$$

In the next section we make much use of the two soliton solution of (2.13) which is

$$E(x,t) = \left(\frac{E_1^2 - E_2^2}{E_1^2 + E_2^2}\right)\frac{E_1 \operatorname{sech}\theta_1 + E_2 \operatorname{sech}\theta_2}{\{1 - B_{12}(\tanh\theta_1 \tanh\theta_2 - \operatorname{sech}\theta_1 \operatorname{sech}\theta_2)\}} \tag{2.17}$$

where $B_{12} = 2E_1E_2/(E_1^2 + E_2^2)$. This has the same form as Lamb's two soliton solution of the SIT equations, as we would expect. Only the θ_i are different:

$$\theta_i = \tfrac{1}{2}E_i\left[t - x\left\langle 1 + \frac{4\alpha_a\omega'_s}{E_i^2 + 4(\omega'_s)^2}\right\rangle\right]_{i=1,2}. \tag{2.18}$$

In § 3 we shall for simplicity mainly treat the sharp line version, in which case (2.18) becomes

$$\theta_i = \tfrac{1}{2}E_i\left\{t - x\left(1 + \frac{4\alpha_a\omega_s}{E_i^2 + 4\omega_s^2}\right)\right\}. \tag{2.19}$$

The broadened solution can be constructed easily by reverting to the frequency ω'_s and integrating as in equation (2.3).

3. The general self-induced transparency pulse

The two soliton solution (2.17) of the RMB equations can be used to obtain a generalization of the self-induced transparency 2π pulse. With E_1 and E_2 *real* in (2.18) this solution

† Another difficulty is that a real dielectric may not manage to accommodate fields of this extreme intensity!

represents the collision of two solitons. By taking E_1 and E_2 to be a pair of antihermitian *complex* constants we obtain a real solution describing a localized pulse with 'internal' oscillations, the exact analogue of the '0π' pulse solution of the SIT equations. Putting $E_1 = -E_2^* = E_0 + 2i\omega_c$, and $\delta_1 = -\delta_2^* = \delta_R + i\delta_I$ in (2.17) and (2.19) gives the following exact solution of the RMB equations:

$$E(x, t) = 2E_0 \operatorname{sech} \theta_R \left(\frac{\cos \theta_I - \gamma \sin \theta_I \tanh \theta_R}{1 + \gamma^2 \sin^2 \theta_I \operatorname{sech}^2 \theta_R} \right) \tag{3.1}$$

where

$$\theta_R = \tfrac{1}{2}E_0(t - m_e x) + \delta_R \tag{3.2a}$$

$$\theta_I = \omega_c(t - m_c x) + \delta_I \tag{3.2b}$$

and $\gamma = \frac{1}{2}E_0/\omega_c$. The two refractive indices m_e and m_c are given by

$$m_e = 1 + \frac{4\alpha_a \omega_s \{E_0^2 + 4(\omega_s^2 + \omega_c^2)\}}{E_0^4 + 8E_0^2(\omega_s^2 + \omega_c^2) + 16(\omega_s^2 - \omega_c^2)^2} \tag{3.3a}$$

$$m_c = 1 + \frac{4\alpha_a \omega_s \{4(\omega_s^2 - \omega_c^2) - E_0^2\}}{E_0^4 + 8E_0^2(\omega_s^2 + \omega_c^2) + 16(\omega_s^2 - \omega_c^2)^2}. \tag{3.3b}$$

As in the 0π solution of the SIT *envelope* equations, the solution (3.1) to the RMB *field* equations has zero time area. Equations (3.1)–(3.3) take on a more familiar look when we choose the arbitrary constants E_0 and ω_c such that $\omega_c = \omega_s$, and $\gamma = E_0/(2\omega_c) \ll 1$. Expanding (3.1) to zeroth order in γ we have

$$E(x, t) = 2E_0 \operatorname{sech}[\tfrac{1}{2}E_0\{t - x(1 + 2\alpha_a \omega_s E_0^{-2})\} + \delta_R] \cos(\omega_s t - \kappa_s x + \delta_I). \tag{3.4}$$

In our dimensionless system of units, (3.4) is exactly the sharp line version of the 2π SIT envelope solution (2.10) modulating a resonant carrier wave. The broadened version of (3.4) gives the broadened SIT solution in the same way as $\gamma \to 0$. (The factor of 2 and the different α arise from the different choice of units.) We thus have the surprising result that the 2π SIT pulse is the limiting case of the 0π RMB pulse. The ratio, γ, of the spectral width of the pulse to its carrier wave frequency is 10^{-3} for a picosecond pulse, so (3.4) is a good approximation to (3.1) in this region.

Given the low density condition, $n \lesssim 10^{18}$ atoms cm^{-3}, the *exact* solution (3.1) is valid up to much higher energies (and shorter pulse lengths), both on and off resonance, and is therefore the generalization of the 2π SIT pulse in those regions where the theory of McCall and Hahn is no longer valid. By expanding (3.1) in a power series in γ we can estimate the order of magnitude of the corrections required in the picosecond and subpicosecond region.

We have in equation (3.4) chosen $\omega_c = \omega_s$. More generally ω_c is an arbitrary constant, not necessarily equal to ω_s, and can be interpreted as the frequency of the on- or off-resonance carrier wave. In what follows we shall not assume that ω_c is resonant.

To first order in γ we can write (3.1) as

$$E(x, t) = 2E_0 \operatorname{sech} \theta_R \cos(\theta_I + \phi(x, t)) \tag{3.5a}$$

$$\phi(x, t) = \gamma \tanh \theta_R. \tag{3.5b}$$

The form of (3.5) represents a 2π pulse which is 'chirped'. Defining the chirping as $\Delta\omega_{ch} = \phi_t$ we find the chirp frequency $\Delta\omega_{ch}/\omega_s = \gamma^2 \operatorname{sech}^2 \theta_R$. This chirping is of second order in γ and is of order 10^{-6} in the picosecond region.

A chirping of this magnitude is hardly of experimental interest, but is of theoretical interest since this small effect does not occur in more approximate theories (Matulic and Eberly 1972).

It is possible to calculate the microscopic polarization $r(x, t)$ and the atomic inversion $u(x, t)$ from our knowledge of $E(x, t)$. To first order in γ this gives the results one would expect from SIT theory, namely

$$r(x, t) = P(x, t)\sin\theta_{\mathrm{I}} + Q(x, t)\cos\theta_{\mathrm{I}} \tag{3.6}$$

where

$$P(x, t) = +\frac{2E_0^2}{4(\Delta\omega)^2 + E_0^2}\tanh\theta_{\mathrm{R}}\,\mathrm{sech}\,\theta_{\mathrm{R}} \tag{3.7a}$$

and

$$Q(x, t) = \frac{2E_0\Delta\omega}{4(\Delta\omega)^2 + E_0^2}\,\mathrm{sech}\,\theta_{\mathrm{R}} \tag{3.7b}$$

$$u(x, t) = -1 + \frac{2E_0^2}{4(\Delta\omega)^2 + E_0^2}\,\mathrm{sech}^2\,\theta_{\mathrm{R}}. \tag{3.8}$$

Here we are near resonance such that $\Delta\omega = \omega_s - \omega_c$. Broadening (3.7), (3.8) merely changes the θ_{R}, and $\Delta\omega \to \Delta\omega' = \omega_s' - \omega_c$. The higher order corrections to (3.6)–(3.8) can easily be calculated but they are unimportant in the picosecond region. Perhaps the most interesting result is that the second harmonic terms in $u(x, t)$ appear only in these higher order terms. It would seem that, in the picosecond region at least, the assumptions (i)–(vi) used in deriving the SIT equations are extremely accurate. This result is rather surprising mathematically, considering the strong nonlinearity of the soliton solutions.

The refractive indices of the envelope (m_{e}) and the carrier wave (m_{c}) can be calculated in special limits from (3.3a) and (3.3b). On resonance to first order in γ we have the sharp line values

$$m_{\mathrm{e}} = 1 + 2\alpha_{\mathrm{a}}\omega_s E_0^{-2} \tag{3.9a}$$

$$m_{\mathrm{c}} = 1 - \tfrac{1}{4}\alpha_{\mathrm{a}}\omega_s^{-1}. \tag{3.9b}$$

In our dimensionless units (3.9a) is the usual sharp line result for the SIT pulse, but (3.9b) is rather surprising since it shows the resonant carrier wave travels slightly faster than c. (Remember $\alpha_{\mathrm{a}} \ll 1$.) Just off resonance we have in the sharp line case for small $\Delta\omega$

$$m_{\mathrm{e}} = 1 + \frac{2\alpha_{\mathrm{a}}\omega_s}{4(\Delta\omega)^2 + E_0^2} \tag{3.10a}$$

$$m_{\mathrm{c}} = 1 + \frac{2\alpha_{\mathrm{a}}\omega_s(2\Delta\omega\omega_s^{-1} - \frac{1}{4}E_0^2\omega_s^{-2})}{4(\Delta\omega)^2 + E_0^2}. \tag{3.10b}$$

The broadened versions of (3.10) are

$$m_{\mathrm{e}} = 1 + \left\langle \frac{2\alpha_{\mathrm{a}}\omega_s}{4(\Delta\omega')^2 + E_0^2} \right\rangle \tag{3.11a}$$

$$m_{\mathrm{c}} = 1 + 2\alpha_{\mathrm{a}}\left\langle \frac{2\Delta\omega' - \frac{1}{4}E_0^2\omega_s^{-1}}{4(\Delta\omega')^2 + E_0^2} \right\rangle. \tag{3.11b}$$

The integration in (3.11) is over $\Delta\omega' = \omega_s' - \omega_c$. If $g(\Delta\omega')$ is symmetric the first term in the numerator in (3.11*b*) will not contribute and m_c will be less than unity. Basically the same results were obtained by McCall and Hahn (1969). Our small correction to McCall and Hahn's theory lies in the second term in the numerator in (3.11*b*) and in higher terms in γ and $\Delta\omega'$ given by the exact forms for m_e and m_c in equations (3.3).

Well off resonance, such that $\Delta\omega \gg E_0$, m_e and m_c in equations (3.10) or (3.11) reduce to the usual refractive index and inverse group velocity of linear theory. This result was first derived in the SIT approximation by Courtens and Szöke (1968).

4. Stability considerations

Lamb (1971) showed that soliton solutions of the SIT equations were stable but not asymptotically stable in the Liapunov sense. We can easily show that our multisoliton solutions of the RMB equations have the same stability property by considering the conservation of energy equation. In fact it is simple to show that *any* solution of the RMB equations satisfying the boundary conditions given in § 2.1 is stable.

The conservation of energy equation follows from (2.13*a*) on multiplication by E and the use of (2.13*b, d*).

$$\frac{\partial}{\partial t}(\tfrac{1}{2}E^2 + \alpha_a\langle\omega_s'(u+1)\rangle) + \frac{\partial}{\partial x}(\tfrac{1}{2}E^2) = 0. \tag{4.1}$$

We have added the constant unity to u to make the hamiltonian density

$$H(x, t) = \tfrac{1}{2}E^2 + \alpha_a\langle\omega_s'(u+1)\rangle \tag{4.2}$$

positive definite. Because of our boundary conditions $H \to 0$ as $x \to \pm\infty$. Next we define the Liapunov functional

$$L(t) = \int_{-\infty}^{+\infty} H(x, t)\,\mathrm{d}x. \tag{4.3}$$

In order to consider the stability properties of a solution, we need the derivative L_t. It follows from (4.2), (4.3) that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \int_{-\infty}^{+\infty} (EE_t + \alpha_a\langle\omega_s' u_t\rangle)\,\mathrm{d}x. \tag{4.4}$$

Using equations (2.13) it is simple to show that

$$\alpha_a\langle\omega_s' u_t\rangle = -E(E_t + E_x) \tag{4.5}$$

so

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\int_{-\infty}^{+\infty} EE_x\,\mathrm{d}x \tag{4.6a}$$

$$= -[\tfrac{1}{2}E^2]_{-\infty}^{+\infty}. \tag{4.6b}$$

Since $E \to 0$ as $x \to \pm\infty$ according to the boundary conditions, $L_t = 0$. This means that any solution which satisfies the boundary conditions, and in particular our N soliton solution, is stable but not asymptotically stable, and small perturbations remain finite. This kind of stability is the best we can expect in a frictionless system (Benjamin 1972). Because of the mathematical correspondence between the SIT and the RMB equations the same results hold for the SIT equations.

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